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LETTER TO THE EDITOR

A class of null solutions to Yang–Mills equations

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Abstract. A class of null solutions of Maxwell's equations is generalised to Yang–Mills theory. The plane-fronted, non-Abelian waves, recently found by Coleman, are among the null solutions.

Consider a plane electromagnetic wave, propagating in the z direction in Minkowski space. Its electric and magnetic fields are

$$\boldsymbol{E} = a\boldsymbol{e}_x + b\boldsymbol{e}_y, \qquad \boldsymbol{B} = a\boldsymbol{e}_y - b\boldsymbol{e}_x, \tag{1}$$

where a and b are arbitrary functions of $u = t - z$, $\boldsymbol{e}_x = \text{grad}\, x$ and $\boldsymbol{e}_y = \text{grad}\, y$. This field may be derived from a potential (A_μ), $\mu = 1, 2, 3, 4$, which will be represented by the 1-form $A = A_\mu\, \mathrm{d}x^\mu$. A possible choice is

$$A = (ax + by + c)\, \mathrm{d}u \tag{2}$$

where c is another function of u. Indeed, the electromagnetic field is given by the 2-form

$$F = (a\, \mathrm{d}x + b\, \mathrm{d}y) \wedge \mathrm{d}u. \tag{3}$$

Clearly, the function c occurring in (2) may be eliminated by a gauge transformation without affecting either a or b.

The potential (2) is suitable for a generalisation to non-Abelian gauge fields, including gravitation, and to a class of null 'spherical' waves.

Consider the line element

$$2\, \mathrm{d}u(\mathrm{d}v + H\, \mathrm{d}u) - P^2(\mathrm{d}x^2 + \mathrm{d}y^2) \tag{4}$$

on a four-dimensional manifold M, referred to coordinates x, y, u and v. It includes Minkowski space in various ways, i.e. in different coordinate systems. In particular, if

$$P = 1 \qquad H = 0 \qquad u = t - z \qquad v = \tfrac{1}{2}(t + z) \tag{5a}$$

then the line element reduces to

$$\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{5b}$$

Similarly, if

$$P = v[1 + \tfrac{1}{4}(x^2 + y^2)]^{-1} \qquad H = \tfrac{1}{2} \qquad u = t - r \qquad v = r \tag{6a}$$

$$x + \mathrm{i}y = 2e^{\mathrm{i}\varphi} \cot \tfrac{1}{2}\vartheta$$

then the line element is

$$\mathrm{d}r^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2 \vartheta\, \mathrm{d}\varphi^2). \tag{6b}$$

The line element (4), considered by Ivor Robinson as early as 1956, also includes the Schwarzschild solution ($H=\frac{1}{2}-m/r$, other identifications as in (6a)), plane gravitational waves, and many other solutions of Einstein's equations (Robinson and Trautman 1962).

Let A be the potential of a gauge configuration in a theory of the Yang–Mills type, based on a Lie group G. The potential is a 1-form on M, with values in $\mathfrak{g}$, the Lie algebra of G. The gauge field corresponding to A is

$$F=\mathrm{d}A+\tfrac{1}{2}[A, A]$$

where the bracket denotes both the Lie algebra product and the exterior (wedge) product of forms. Introducing the Hodge (dual) operator $*$ one can write source-free Yang–Mills equations as

$$D*F \underset{\text{def}}{=} d*F+[A, *F]=0. \tag{7}$$

Assuming the metric on M to be of the form (4) and orientation to be given by the coframe θ,

$$\theta' = P\,\mathrm{d}x \qquad \theta^2=P\,\mathrm{d}y \qquad \theta^3=\mathrm{d}u \qquad \theta^4=\mathrm{d}v+H\,\mathrm{d}u, \tag{8}$$

the duals of 2-forms may be evaluated from

$$*(\mathrm{d}u\wedge\mathrm{d}x)=\mathrm{d}u\wedge\mathrm{d}y \qquad *(\mathrm{d}u\wedge\mathrm{d}y)=-\mathrm{d}u\wedge\mathrm{d}x \qquad \text{etc.}$$

Let the potential be

$$A=\alpha(x, y, u)\,\mathrm{d}u$$

where α is a $\mathfrak{g}$-valued function of the variables indicated. The bracket $[A, A]$ vanishes because $\mathrm{d}u\wedge\mathrm{d}u=0$. Therefore

$$F=(\alpha_x\,\mathrm{d}x+\alpha_y\,\mathrm{d}y)\wedge\mathrm{d}u$$

and

$$*F=(\alpha_x\,\mathrm{d}y-\alpha_y\,\mathrm{d}x)\wedge\mathrm{d}u$$

where subscripts x and y denote partial derivatives with respect to the variables indicated.

Since $[A, *F]=0$, the Yang–Mills equation (7) reduces to

$$\alpha_{xx}+\alpha_{yy}=0$$

so that the general solution is $\alpha=\mathrm{Re}\,w\,(x+\mathrm{i}y, u)$, where w is an arbitrary $\mathfrak{g}$-valued function of u, analytic in $x+\mathrm{i}y$.

In the coordinate system (5), the solution

$$\alpha=a(u)x+b(u)y+c(u)$$

is a 'non-Abelian plane wave' (Coleman 1977). But is it really plane, i.e. invariant under translations in the x and y directions? By looking at the field, given by formula (3), one is tempted to give an affirmative answer to this question. It should be remembered, however, that in a non-Abelian gauge theory, F does not provide a complete description of the gauge configuration.

A transformation (diffeomorphism) f of M is a symmetry of a gauge configuration if the transformed (pulled-back) potential f^*A differs from A only by a change of gauge, i.e. if there is a G-valued function g on M such that

$$f^*A = g^{-1}Ag + g^{-1}\,\mathrm{d}g. \tag{9}$$

An infinitesimal version of the condition of invariance,

$$\mathscr{L}_\xi A = D\eta, \tag{10}$$

is obtained from (9) by considering the one-parameter groups (f_t), (g_t), $t \in \boldsymbol{R}$, where (f_t) is generated by a vector field ξ on M and $g_t = \exp t\eta$ with $\eta : M \to \mathfrak{g}$ (Bergmann and Flaherty 1978, Trautman 1979). Differentiation of both sides of equation (10) leads to

$$\mathscr{L}_\xi F = [F, \eta] \tag{11}$$

where, as in (10), $\mathscr{L}_\xi$ is the Lie derivative with respect to ξ.

If ξ is a generator of a translation in the (x, y) plane, say $\xi = \partial/\partial x$, then equation (11) may be satisfied for a field of the form (3), but equation (10), in general, cannot. Indeed, let $G = \mathrm{SO}(3)$, so that $\mathfrak{g}$ is isomorphic to $\boldsymbol{R}^3$ and the Lie bracket corresponds to the vector product. Equation (11) implies

$$[a, \eta] = 0 = [b, \eta] \tag{12}$$

and equation (9) reduces to

$$a\,\mathrm{d}u = \mathrm{d}\eta + [c, \eta]\,\mathrm{d}u. \tag{13}$$

If $[a, b] \neq 0$, then conditions (12) imply $\eta = 0$, in contradiction to (13). Moreover, the function c cannot be changed by a gauge transformation without affecting a and b. Therefore, in an SO(3) theory, two potentials of the form (2), with different functions c, and the same, but non-commuting a and b, are gauge-inequivalent.

If the values of a, b and c belong to an Abelian Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then c can be reduced to 0 without changing either a or b; in this case translations in the (x, y) plane are symmetries.

Plane-fronted gravitational waves may be described by the line element (4) with $P = 1$ and a linear connection of the form

$$\omega = \alpha\,\mathrm{d}u \tag{14}$$

when referred to the frame (8); α is now a $\mathfrak{gl}(4, \boldsymbol{R})$-valued function of x, y and u, $\alpha = (\alpha^\mu{}_\nu)$. The connection is metric if and only if

$$\alpha_{\mu\nu} + \alpha_{\nu\mu} = 0.$$

It is moreover torsionless if the only non-vanishing elements of the matrix $(\alpha_{\mu\nu})$ are

$$\alpha_{31} = -\alpha_{13} = H_x \qquad \alpha_{32} = -\alpha_{23} = H_y.$$

In other words, the conditions of symmetry and metric compatibility, imposed on the connection 1-form (14), imply that the values of α belong to the *Abelian*, two-dimensional Lie algebra $\mathfrak{h}$ of the group of null Lorentz transformations which leave invariant the null vector $\partial/\partial v$ (see, for example, Ehlers *et al* 1966).

The only essential components of the curvature two-form

$$\Omega^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$$

are

$$\Omega_{31} = (H_{xx}\,\mathrm{d}x + H_{xy}\,\mathrm{d}y) \wedge \mathrm{d}u$$

$$\Omega_{32} = (H_{xy}\,\mathrm{d}x + H_{yy}\,\mathrm{d}y) \wedge \mathrm{d}u.$$

The Einstein equation for empty space

$$\theta_{[\mu} \wedge \Omega_{\nu\rho]} = 0$$

reduces to

$$H_{xx} + H_{yy} = 0,$$

whereas the Kilmister–Yang equation (Kilmister 1962, Yang 1974)

$$D * \Omega_{\mu\nu} = 0$$

is equivalent to the weaker condition

$$H_{xx} + H_{yy} = \text{arbitrary function of } u.$$

Both of these field equations admit plane waves as solutions. A pure gravitational plane-wave solution of Einstein's equations is given by (Bonnor 1956, Peres 1959)

$$H = \mathrm{Re}\,\{w(u)(x + \mathrm{i}y)^2\}.$$

The corresponding metric is plane because it admits a group of isometries transitive on the null hyperplanes u = constant (Bondi *et al* 1959, Ehlers and Kundt 1962).

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